Sparse approximations and compressed sensing: an overview

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A revolution in sampling theory

- During the last 6 years, we have been witnessing a revolution in sampling theory.
- Initiated by the works of Donoho (plenary talk on Monday at the CMS meeting) and of Candès and Tao (Fields medalist), around 2004.
- Opened up a new field called compressed sensing or compressive sampling: Very active area. “Special Session on Compressed Sensing” at CMS Winter Meeting co-organized by Friedlander, Herrmann, OY. BIRS Workshop in March co-organized by OY.
- Relies heavily on the theory of sparse approximations that has been around for more than two decades (transforms such as wavelets, curvelets, Gabor,…).
- Various projects under DNOISE II (aim to) leverage and improve these new techniques.
Motivation: Signal Acquisition and Processing

**Inherently analog signals:** Audio, images, seismic, etc.

**Objective:** Use digital technology to store and process analog signals – find efficient digital representation of analog signals.
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How is this done - classical approach

Signal $f$ (analog) → Sampling (I) → Quantization (II) → Compression (III)

A/D conversion: measurement & truncation
Source coding: truncation & compression (or other processing)
Stage I (Sampling)

- samples obtained on a dense temporal/spatial grid,
- an appropriate sampling theorem ties resolution of “reconstruction” with the grid density.
Classical Approach

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Example: Shannon-Nyquist Sampling Theorem.

Suppose \( f \) is bandlimited with bandlimit \( \Omega \), i.e., \( f \in B_\Omega \). Then for \( \tau < \frac{1}{2\Omega} \), we have

\[
f(t) = \sum_n f(n\tau) \phi(t - n\tau), \quad \forall t.
\]

Above \( \phi \) can be chosen with fast decay so the reconstruction is local.
Classical Sampling Theorem: The picture

A bandlimited $f$

Fourier transform of $f$
Classical Sampling Theorem: The picture

A bandlimited $f$

Fourier transform of $f$

Need $N \approx 2\Omega \times 2T$ samples to reconstruct $f$ on $[-T, T]$.

Equivalently: Every bandlimited function $f \in B_{\Omega}$, on $[-T, T]$ can be represented by a vector $f \in \mathbb{R}^N$ which we obtain by collecting $N$ measurements.
What makes the classical sampling approach work?

1. \( f \), the signal of interest, is structured \( \leftarrow \) “model the signal class”.

2. We measure \( f \) by obtaining its samples on a regular grid \( \leftarrow \) “specify the measurement scheme”.

3. Use Shannon-Nyquist sampling theorem to reconstruct \( \leftarrow \) “find a reconstruction method”.

Note that:

- ambient dimension of the corresponding representation is \( N \sim \Omega T \).

- Different \( N \)-dimensional vectors correspond to samples of different bandlimited functions – so **no hope for dimension reduction**—i.e., we need \( N \) independent measurements—under this signal model.
**Compressive Sampling Theory**

**Above:** Reduced a bandlimited function $f$ to a vector $f$ in $\mathbb{R}^N$.

**Question:** Can we reduce the dimensionality of the problem by restricting the signal class further?

Another bandlimited $f$

![Another bandlimited $f$](image)

Fourier transform of $f$

![Fourier transform of $f$](image)

An additional constraint of $f$: its **Fourier transform is sparse**.

Do we still need $N \approx 4\Omega T$ samples to reconstruct $f \in \mathbb{R}^N$?
**Compressive Sampling Theory**

**Rephrase the question:** Suppose we know that

\[ f \in B_\Omega \text{ and } f \text{ has a sparse Fourier transform.} \]

Do we still need to sample at the Nyquist rate for a good (perfect) reconstruction?

Consider the following set of samples (at irregular points):

![Graph showing irregularly spaced samples](image)

Here: average sampling density is only 50% of Nyquist rate.

**Claim:** We can recover \( f \) from these samples!
Given:

1. Fourier transform of $f$ is sparse.
2. We only know a few irregular samples, say $n$, that we showed in the previous slide. Using linear algebra, write:

   $$f_{\text{samples}} = Rf$$

   where $R$ is an $n \times N$ “restriction matrix” (with $n \ll N$).

3. We also know that

   $$f = F^*x, \text{ where } x \text{ is the Fourier transform of } f.$$ 

Combining these:

$$f_{\text{samples}} = RF^*x, \text{ where the only unknown is } x.$$ 

Can we solve for $x$? If yes, we recover $f$ via $f = F^*x$. 

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Recall: We have $f_{\text{samples}} = \left( RF^* \right) A x$ which we want to solve for $x$.

Notes:

- $A = RF^*$ is an $n \times N$ (short) matrix – i.e., the system is underdetermined and has infinitely many solutions.
- However, we also know that the solution we seek is the Fourier transform of $f$, thus sparse.
- So, how about the “sparsest” solution of the system above?

Solve

$$x_{\text{approx}} = \arg \min \| z \|_0 \text{ subject to } Az = f_{\text{samples}}.$$
Compressive Sampling Theory – imposing sparsity

Here is the reconstruction obtained from the above samples (approx. 50% of Nyquist rate)

- We get essentially perfect reconstruction!
- How did we solve the combinatorial optimization problem:

\[ \min \|z\|_0 \text{ subject to } Az = f_{\text{samples}} \]

We will come back to this later.
Compressive Sampling Theory – general framework

- Signal \( f \in \mathbb{R}^N \), want to collect information on \( f \).
- **Model the signal class:** \( f \) admits a sparse representation w.r.t. a known basis \( B \): \( f = B^*x \) where \( x \) is sparse.
- **Specify a measurement scheme:** Construct an \( m \times N \) measurement matrix \( M \) with \( m \ll N \) (above this was the restriction matrix \( R \))
  \[
  f_{\text{meas}} = Mf = MB^*x
  \]
- **Reconstruction method:** Solve the underdetermined sparse recovery problem:
  \[
  x_{\text{approx}} = \text{“sparsest” } z \text{ such that } f_{\text{meas}} = MB^*z.
  \]
Compressive Sampling Theory: main questions

Sparse recovery problem:
\[ x_{\text{approx}} = \text{"sparsest"} \ z \text{ such that } f_{\text{meas}} = MB^*z. \]

Main questions:

1. How do we find the sparsifying basis \( B \)?
2. How do we construct the measurement matrix \( M \)?
3. How many measurements do we need to have \( x_{\text{approx}} = x \)?
4. How do we solve the sparse recovery problem?
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First address question 1: How do we find sparsity transforms?

- Note that this is dependent heavily on the class of signals of interest.
- In the above example, the sparsity transform was Fourier transform.
- Applied and computational harmonic analysis community has been developing such transforms during the last three decades that are tailored to important signal classes such as: audio, natural images, seismic data and images.
- Rich area with interesting mathematics, directly applicable constructive results such as wavelet transform, curvelet transform etc.
- Next, we give examples of some important sparsity transforms.
Wavelet transform sparsifies natural images.
Short-time Fourier (Gabor) transform sparsifies audio signals.

Audio signal

A Gabor atom

STFT transform

Sorted coefficients
Curvelet transform sparsifies seismic data and images.

sampled Green’s function

a curvelet atom

sorted coefficients
Sparse recovery problem:

\[ x_{\text{approx}} = \text{“sparsest” } z \text{ such that } f_{\text{meas}} = MB^*z. \]

Main questions:

1. How do we find the sparsifying basis \( B \)?
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Sparse recovery problem

Fix a sparsity basis $B$ and a measurement matrix $M$ (more later). Set $A = MB^*$. We need to solve:

Recall: $A$ is $n \times N$, $n \ll N$. We are after reconstruction algorithms, i.e., decoders, $\Delta : \mathbb{R}^n \mapsto \mathbb{R}^N$, with the following properties:

**C1.** $\Delta(Ax) = x$ whenever $x$ is $k$-sparse (exact reconstruction for sufficiently small $k$).

**C2.** $\|x - \Delta(Ax + e)\| \lesssim \|e\| + \|x - x_k\|$. Here $e$: measurement error, e.g., thermal and computational noise. Reconstruction works for noisy measurements and approx. sparse signals.

**C3.** $\Delta(\cdot)$ can be computed efficiently (in some sense).

Whether we can achieve C1–C3 will depend on the choice of the measurement matrix $A$ and dimensionality relations between $n$, $N$, and $k$. 

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Sparse recovery problem – the decoder $\Delta_0$

Given $y = Ax$, the (noise-free) encoding of $x$, we want to find $x$. Clearly, this problem is non-trivial:

- **underdetermined** system $y = Az$ has **infinitely** many solutions (provided $A$ is full-rank).
- $x$ is one of these! Decoder must **choose the correct solution**.
- An intuitive decoder: choose the sparsest solution.

$$\Delta_0(b) := \arg \min_z \|z\|_0 \text{ subject to } y = Az.$$
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**Theorem (Donoho et al.)**

If $A$ is in general position (i.e., its Kruskal rank or its “spark” is $n$), then $\Delta_0(Ax) = x$ for $x \in \Sigma^N_s$ with $s < n/2$.

**Note:** $\Delta_0$ is not stable or robust. More importantly, we need to solve a combinatorial optimization problem.
The optimization problem for $\Delta_0$ is combinatorial. Need alternatives.

**How about $\ell_2$ minimization?** Choose the solution with smallest 2-norm:

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$$\Delta_2 = A^*(AA^*)^{-1}.$$ The solution is not sparse.
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**How about $\ell_2$ minimization?** Choose the solution with smallest 2-norm:

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**A much better alternative: $\ell_1$ minimization.** Choose the solution with smallest 1-norm:

$$\Delta_1(y) := \arg\min_z \|z\|_1 \text{ subject to } y = Az.$$ 

This can be formulated as a convex program. Moreover, unlike 2-norm, 1-norm promotes sparsity. (See talks by M. Friedlander and T. Lin.)
Sparse recovery by 1-norm minimization

Recent exciting developments show that $\Delta_1$ satisfies the conditions (C1)-(C3), thus “equivalent to $\Delta_0$”, under certain conditions.

**Theorem (Candès-Romberg-Tao, Donoho)**

*Suppose that $A$ is “sufficiently similar” to an orthonormal matrix. Then, there exists $k_{\text{max}}(A)$ such that*

$$
\|x - \Delta_1(Ax + e)\|_2 \lesssim \|e\|_2 + k^{-1/2} \|x - x_k\|_1
$$

*for all $k \leq k_{\text{max}}$. In particular,*

$$
x \text{ is } k\text{-sparse} \Rightarrow \Delta_1(Ax) = x.
$$

**Remark.** $\Delta_1$ satisfies (C1)-(C3) if $x$ is sufficiently sparse. Next, we investigate the dependence of $k_{\text{max}}$ to $A$. 
How to choose the measurement matrix

- There are precise conditions on $A$ (in terms of its RIP constants) that guarantee that the theorem holds.

- For example, if $A$ is a random matrix with iid Gaussian entries, then

$$n \gtrsim k \log(N/k)$$

will suffice. **Number of measurements scale only logarithmically with the ambient dimension: grid size in our previous example.**

- This is theoretically optimal (deep results in geometric functional analysis).

- Other classes (Bernoulli, partial Fourier, ...) of random matrices will do, too!
How to choose the measurement matrix — more remarks

- Gaussian and sub-Gaussian matrices are unitarily invariant, so the dimension relation is independent of the sparsity basis. These are universal measurement matrices:

  \[ M \text{ is Gaussian and } B \text{ is unitary } \implies A = MB^* \text{ is Gaussian.} \]

- Ideal for dimension reduction in simulations. Also, acquisition with simultaneous sources.

- Difficult to implement depending on the physics—e.g., in the sampling example. In such cases:
  - sample in a domain that is incoherent with the sparsity domain: e.g.,
    
    \[ \text{sparse in Fourier } \implies \text{sample in time} \]
  - Randomly sub-sample (possibly on a jittered grid), i.e., “apply” a restriction matrix \( R \).

The corresponding \( A = RM \) will be a “good” compressive sampling matrix.
Impact on applications

One of the main conclusions of compressive sampling theory: *Accuracy scales logarithmically with the grid size.*

Far-reaching implications:

- Digital camera technology (e.g., Baraniuk et al.)
- Medical imaging (e.g., Lustig and Donoho)
- Analog-to-digital conversion (e.g., Baraniuk et al.,)
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- **Seismic imaging and interpolation (SLIM)**
  - Denoising
  - Interpolation – recovery from data with missing traces
  - Primary/multiple separation
  - Acquisition with simultaneous sources
  - “Compressed computation” and full wave inversion
  - ...
Our theoretical contributions

Past work:

- CS recovery algorithms via non-convex optimization (Saab and Y)
- A provably convergent algorithm for coherent source separation (Wang, Saab, Herrmann, Y)
- Efficient quantization (for A/D conversion) of compressive samples (Saab, Y, and collaborators)
- Improved compressive recovery when partial, relatively accurate support information is available via weighted $\ell_1$ minimization (Mansour, Saab, Friedlander, Y) – see Mansour’s talk.

Ongoing work:

- Leveraging our recent results on weighted $\ell_1$ to obtain a “rewighted” version with provable recovery guarantees (Mansour).
- Leverage higher-dimensional structure in the sparsity pattern – measurement matrices that can be factored as Kroenecker products (Saab – see Saab’s talk)
- …
Compressive sampling theory: number of samples scales only logarithmically with the grid size!

Theory helps us design effective (optimal) acquisition geometries.

Dimension reduction process is linear; reconstruction is non-linear. (In contrast to classical methods where reconstruction is linear, however dimension reduction is non-linear.)

Transforming consequences for seismic (as well as other) signal acquisition and processing.

Very active area of research — 100s of papers in the last few years.

Active group at UBC, covering this area from all angles: theory, algorithms, and applications to various problems in exploration seismology – see the upcoming talks.
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