Pure P-wave equations for acoustic modeling in transversely isotropic (TI) media are derived by approximating the exact pure P-wave dispersion relation. We present an alternative approach to the common Taylor expansion based equations, in which the acoustic wave equation is approximated by a polynomial expansion and determine its coefficients by solving a least squares problem that minimizes the phase velocity error over a defined range of Thomsen parameters. Further approximations can be used to reduce the computational cost. For a TI medium, one time step of our equation can be calculated at the cost of 22 FFTs, which is the same number that is required for the standard pure P-wave equation. For 3D TI, we require 66 FFTs.

Figure 1: Relative phase velocity error for two different TI media and various pure P-wave equations.

Figure 2: Absolute maximum phase velocity error over a range of Thomsen parameters for the standard pure P-wave equation (white) and our new optimized scheme (black).

Figure 3: Snapshots of wavefields after propagating 300 wavelength at 0.4 in a constant VTI medium with ε = 0.63 and φ = 0.05.

Conclusion
We developed a new scheme for pure P-wave modeling in VTI and TI media based on polynomial expansions. For a fixed number of expansion terms, our equation has the minimum phase velocity error, which is a measure of error in a certain range of Thomsen parameters. We show that our equations is up to an order of magnitude more accurate than other pure P-wave equations and the computational cost can be limited through a recursive calculation of the terms.

Acknowledgements
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\[ \alpha_i = \arg \min \left( \sum_i \left( \frac{1}{\epsilon_i} \right) \left( 1 - \frac{E_i(\alpha_j, \alpha_k)}{E_i(0, 0)} \right)^2 \right) \]

where \( \alpha_i \) and \( \alpha_k \) are the determined such that the minimum relative phase velocity error of our scheme. That is, let \( \alpha_i = 0 \) and using (\( c_1 = \epsilon_i \)), and determine the coefficients by solving a least squares problem that minimizes the phase velocity error of our scheme.

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Extension to spatially varying anisotropy

The coefficients \( c_{j,k} \) for different values of \( j \) and \( k \) are obtained for a particular value of \( n \) and \( m \), however they vary slightly for varying Thomsen parameters, which makes it possible to interpolate them:

\[ c_{j,k} = c_{j,k}(n, m) \]

where \( c_{j,k} \) are Lagrange polynomials of the corresponding order. In this phase velocity error minimizing scheme using 2D phases between 0 and 2π radians of the coefficients, but not only optimized within a range of phase angles, as well as several range of Thomsen parameter.

Forward modeling scheme

For the TI medium, our optimized scheme can be directly translated to a forward modeling scheme:

\[ \epsilon = 0.05 \]

\( |n| \leq 0.5 \) and \( |m| \leq 0.5 \).

The function \( f_{j,k} \) is a sine function using 2D phases between 0 and 2π radians of the coefficients, but not only optimized within a range of phase angles, as well as several range of Thomsen parameter.

\[ f_{j,k} = f_{j,k}(n, m) \]

where \( f_{j,k} \) are Lagrange polynomials of the corresponding order. In this phase velocity error minimizing scheme, we use Legendre polynomials up to order 3, which means we solve for all anisotropic parameters. \( P(1, \ldots, 1) = 1 \).

Phase velocity error analysis

We compare the relative phase velocity error of our optimized pure P-wave equation with the errors of the "imperfect" pure P-wave equation, as well as with the Chu et al. equation with the geometry's square \( v = 1 \) and \( j = n \) = \( m \) = 0.05.

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Phase velocity error minimizing pure P-wave equation

Instead of a Taylor expansion, we approximate the exact pure P-wave dispersion relation using a polynomial expansion:

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \]

where the spatially dependent coefficients \( a_{j,k}(x, y, z) \) are determined such that the minimum relative phase velocity error of our scheme.

\[ a_{j,k}(x, y, z) = \alpha_j \left( \frac{k_x^2 \alpha_k \alpha_j}{\epsilon_i} - \frac{k_y^2 \alpha_k \alpha_j}{\epsilon_i} + \frac{k_z^2 \alpha_k \alpha_j}{\epsilon_i} \right) \]

and the relative phase error as a function of \( n \) and \( m \):

\[ E(n, m) = \left( \frac{1}{\epsilon_i} \right) \left( 1 - \frac{E_i(\alpha_j, \alpha_k)}{E_i(0, 0)} \right)^2 \]

where \( \epsilon_i \) is the error of the phase velocity of our scheme.

\[ E_i(\alpha_j, \alpha_k) = \left( \frac{1}{\epsilon_i} \right) \left( 1 - \frac{E_i(0, 0)}{E_i(0, 0)} \right)^2 \]

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