Seismic wavefield inversion with curvelet-domain sparsity promotion

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General statement

• Recent resurgence of wavefield inversions
  – *imaging* where the ‘sunken’ source & data-residue wavefields are inverted [Claerbout, Berkout and others]
  – *focal transform* where primaries are deconvolved to focus data [Berkhout ‘06]
  – *interferometric deconvolution* where wavefields are inverted [Vasconcelos & Snieder ‘08, Wapenaar ‘08]
  – *data inverse* where the data itself is inverted [Berkhout ‘06]

• Challenge is to *stably* invert these *wavefields*
  – in the presence of noise, finite aperture, and source signatures
  – for incomplete & simultaneously acquired data

• Propose a *regularization* based on curvelet-domain sparsity promotion enforced by nonlinear optimization ...
Inverse data-matrix
Problem statement

- Seismic wavefield inversions = multi-D deconvolutions

- Corresponds to the inversion of Berkhout’s [‘82] data matrix
  - monochromatic
  - inverted by damped & weighted least-squares matrix inversion [Wapenaar ‘08]

- Suffers from instabilities that limit applicability to real data
  - noise
  - finite acquisition
  - incomplete data

- Present a framework for stable inversion with sparsity promotion.
Motivation

• Successful application of curvelets
  – wavefield recovery from missing traces [F.J.H & Hennenfent ‘08, Hennenfent & F.J.H ‘08]
  – wavefield recovery from compressive simultaneous simulations [F.J.H et. al ‘08]
  – curvelet-transform [Candes et. al. ‘06] based sparsity promotion

• Robustness & uplift of focused curvelet-based wavefield recovery
  – curvelet-regularized inversion of the primary-data-matrix operator [F.J.H et. al. ‘07-’08]
  – incorporation of a priori information
  – improved wavefield recovery from missing traces

• Insights from compressive sampling [Donoho ‘06, Candes et.al ‘06, Lin & F.J. H ‘07]
  – jittered sampling [Hennenfent & F.J.H]
  – blended-source design [F.J.H et.al ‘08]
  – one-norm solvers [Hennenfent et. al. ‘08]

• Move from multi-D correlations to multi-D deconvolutions ....
2D discrete curvelets
Sparsity-promoting program

Solve for $x_0$

- exploits \textit{sparsity} in the curvelet domain as a \textit{prior}
- finds the sparsest set of curvelet coefficients that match (incomplete) data
- inverts an \textit{underdetermined} system

\[ P_\epsilon : \begin{cases} \tilde{x} = \arg \min_x \| x \|_1 \quad \text{s.t.} \quad \| A\tilde{x} - y \|_2 \leq \epsilon \\ \tilde{g} = S^H \tilde{x} \end{cases} \]

Acquired data

Complete wavefield (transform domain)

Restriction operator

\[ A := RS^H \]

Redundant sparsifying transform

\[ \tilde{x} = \arg \min_x \| x \|_1 \quad \text{s.t.} \quad \| A\tilde{x} - y \|_2 \leq \epsilon \]

\[ \tilde{g} = S^H \tilde{x} \]

Observations:

- exploits \textit{sparsity} in the curvelet domain as a \textit{prior}
- finds the sparsest set of curvelet coefficients that match (incomplete) data
- inverts an \textit{underdetermined} system

\[ \text{Sacchi et al.'98} \]
\[ \text{Xu et al.'05} \]
\[ \text{Zwartjes and Sacchi'07} \]
\[ \text{F.J.H and Hennenfent'08} \]
Data matrix (2D seismic line)
Subsampling by restriction (picking)

For each time-slice along source-receiver coordinates

\[ B = \left( R^{\Sigma_r} \right)^* U \left( R^{\Sigma_s} \right) \]

or more succinctly with Kronecker products

\[ b = \left( R^{\Sigma_s} \otimes R^{\Sigma_r} \right) \text{vec} (U) \]

For all time slices in the data matrix, we have

\[ R = \left( R^{\Sigma_s} \otimes R^{\Sigma_r} \otimes I \right) \]
Incomplete data
Curvelet-domain sparsity promotion
Wavefield recovery by sparsity promotion

\[
\begin{align*}
R &= \left( R^{\Sigma_s} \otimes R^{\Sigma_r} \otimes I \right) \quad \text{(source-receiver restriction)} \\
b &= R \text{vec} (U) \quad \text{(incomplete data)} \\
A &= RS^H \\
\tilde{x} &= \arg \min_x \| x \|_1 \quad \text{s.t.} \quad \| Ax - b \|_2 \leq \epsilon \\
\tilde{U} &= \text{vec}^{-1} \left( S^T \tilde{x} \right) \quad \text{(recovered data)}
\end{align*}
\]

Curvelet sparsity underlies success of wavefield recovery
- from large percentages of traces missing [F.J.H & Hennenfent ‘08]
- improvements from jittered subsampling [Hennenfent & F.J.H ‘08]

Formulation
- only exploits curvelet-domain sparsity
- misses focusing with wavefields

Can we extend this formalism to invert wavefields ....?
Common-problem formulation

• Extension of curvelet-based wavefield recovery to include (de)focusing with data-matrices defined by wavefields [F.J.H et.al ’07-’08]
  – define linear data-matrix operators
  – multi-D convolutions
  – and their adjoint multi-D correlations

• Incorporates prior information

• Use transform-domain sparsity to stably invert for all frequencies

• Combination of sparsity and focusing
Common approach: damped least-squares

**Monochromatic forward model:**

to be inverted wavefield

\[
\hat{G} = \hat{U} \hat{U}^H \left( \hat{U} \hat{U}^H + \epsilon^2 I \right)^{-1} \hat{V}
\]

unknown ”image”

known wavefield

**Monochromatic pseudo-inverse:**

[Berkhout '82]
[F.J.H '07-'08]
[Wapenaar '08]
Curvelet-based wavefield inversion (CWI)

Cast into rigorous *linear-algebra* framework, i.e.

\[ \hat{G}_i \hat{U}_i = \hat{V}_i, \ i = 1 \cdots n_f \]

which with the Kronecker identity

\[ \text{vec} (AXB) = \left( B^H \otimes A \right) \text{vec} (X) \]

becomes for each *frequency*

\[ \left( I \otimes \hat{U}_i \right) \text{vec} (\hat{G}_i) = \text{vec} (\hat{V}_i), \ i = 1 \cdots n_f \]

Set up a system for *all frequencies* and incorporate the *temporal Fourier* transform ....
Curvelet-based wavefield inversion (CWI)

\[
\begin{pmatrix}
F^H \\
I \\
I
\end{pmatrix}
\begin{bmatrix}
I \\
\hat{U}_1 \\
\hat{U}_n_f
\end{bmatrix}
F =
\begin{bmatrix}
G_1 \\
G_{n_f}
\end{bmatrix}
= 
\begin{bmatrix}
V_1 \\
V_{n_f}
\end{bmatrix}
\]

with \( F = (I \otimes I \otimes F) \) (temporal Fourier transform)

Linear system is
- conducive to curvelet-based wavefield inversion with sparsity promotion
- versatile
- conducive to compressive subsampling (e.g. missing trace or blended acquisition)
**Focal transform** [Berkhout ’06, F.J.H et.al ‘07-’08]

\[U = \Delta P\]  
(primary data-matrix operator)

\[V = P\]  
(total data matrix)

\[b = \text{vec}(V)\]

\[A = UC_3^H\]  
(focused 3-D curvelet transform)

\[\tilde{x} = \arg\min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon\]

\[\tilde{G} = \text{vec}^{-1} \left( C_3^H \tilde{x} \right)\]  
(focused data)

- Primary data-matrix operator is inverted
- Total data multi-D deconvolved with the primaries
- Primaries focused to a directional source
- First-order multiples mapped to primaries
Slice from the total data matrix (V)
Slice from primary data-matrix operator (U)
Focused/multi-D deconvolved data (G)
Curvelet-based wavefield inversion (CWI)

\[
P_\epsilon : \begin{cases} 
    b &= \text{vec}(V) \\
    A &= US^H \\
    \tilde{x} &= \arg\min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon \\
    \tilde{G} &= \text{vec}^{-1}\left(S^T \tilde{x}\right) \approx U^\dagger V \\
\end{cases}
\]

Corresponds to

- curvelet-sparsity \textit{regularized} inversion
- multi-D \textit{deconvolution} of the wavefield in the data matrix $U$ with respect to the wavefield in the data matrix $V$

Applications

- \textit{focused} wavefield recovery
- \textit{defocussed} multiple prediction
- data \textit{inverse}
- imaging of \textit{blended} data
Focused wavefield recovery
Motivation

- Exploit *wavefield* focusing in the solution of the *recovery* problem
  - invert subsampled primary data-matrix operator [F.J.H et.al ‘07-’08]
  - interpolate by taking the inverse focal and curvelet transforms

- Combination of sparsity and *wavefield* focusing
  - improved focusing $\Rightarrow$ more sparsity
  - curvelet sparsity $\Rightarrow$ better focusing
Focused wavefield recovery

\[
\begin{align*}
R &= \left( R_{\Sigma_s} \otimes R_{\Sigma_r} \otimes I \right) \quad \text{(source-receiver restriction)} \\
V &= P \quad \text{(total data matrix)} \\
b &= R \text{vec} \left( V \right) \quad \text{(incomplete data)} \\
U &= \Delta P \quad \text{(primary data-matrix operator)} \\
A &= RS^H \\
S^H &= UC^H_3 \quad \text{(focussed 3-D curvelet transform)} \\
\tilde{x} &= \arg \min_x \| x \|_1 \quad \text{s.t.} \quad \| Ax - b \|_2 \leq \epsilon \\
\tilde{V} &= \text{vec}^{-1} \left( S^H \tilde{x} \right) \quad \text{(recovered data)}
\end{align*}
\]

- Restrictions along the source-receiver coordinates
- Focusing by inversion of the restricted primary-data matrix operator
- Reconstruction by inverse curvelet transform and defocusing
Incomplete data
Curvelet-domain sparsity promotion
Focused curvelet-domain sparsity promotion
Defocussed multiple prediction
Motivation

- **Multiple** prediction by multi-D convolution with the primary data-matrix operator
  - requires extensive *matching* to compensate for
    - the “source signature”
    - finite acquisition aperture
    - etc.

- **Defocussed** multiple prediction by multi-D deconvolution with the primary data-matrix operator
  - inversion of the adjoint=multi-D correlation with the primary data-matrix operator
  - compensates for the *amplitudes, finite aperture, & source wavelet*
Defocussed multiple prediction

\[
\begin{align*}
U &= \Delta P^H \\
V &= P \\
b &= \text{vec}(V) \\
A &= US^H \\
S^H &= C^H_3 \\
\tilde{x} &= \arg \min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon \\
\tilde{P} &= \text{vec}^{-1}\left(S^H\tilde{x}\right) \\
\end{align*}
\]

- Defocussing by inversion of the \textit{adjoint} of primary-data matrix operator
- Multi-D \textit{deconvolution} of the multi-D correlation with the \textit{primaries}
- Reconstruction by \textit{inverse} curvelet transform

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Defocussed multiple prediction

multi-D convolution

multi-D deconvolution
Amplitude spectra (averaged)

multi-D convolution

multi-D deconvolution
Stable computation of the ‘data inverse’
Motivation

- **Data-matrix inverse domain** leads to a natural separation of *primaries* and *surface-related multiples* [Berkhout ‘06]

\[
\hat{P}^\dagger = \Delta \hat{P}^\dagger - \hat{A},
\]

- inverted data
- inverted primaries
- ‘source’

- surface-related effects including source signature are mapped to a directional source
- primaries are mapped to the inverse of the primary data matrix

- Application to *real* data hampered by instabilities ...
Data inverse

\[
\begin{align*}
U &= P \quad \text{(total data operator)} \\
V &= I_\Psi \quad \text{(bandwidth-limited delta)} \\
b &= \text{vec}(V) \\
A &= US^H \quad \text{(multi-D convolution)} \\
S^H &= C_3^H \quad \text{(3-D curvelet transform)} \\
\tilde{x} &= \arg \min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon \\
\tilde{P} &= \text{vec}^{-1}\left(S^T\tilde{x}\right) \quad \text{(inverted data)}
\end{align*}
\]

- Inversion of the *full-data matrix operator*
- Multi-D *deconvolution* of the multi-D *convolution* with the data
- Regularized by curvelet-domain *sparsity* promotion
bandwidth-limited pulse

the same in f-k space
Data inverse synthetic data

offset (m)

-1000  0  1000

-0.5

0

0.5

1

1.5

2

time (s)

total data

offset (m)

-1000  0  1000

-1.5

-1

-0.5

0

0.5

1

1.5

2

Time (s)

total data inverse

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Data inverse synthetic data

estimated primaries

estimated-primaries inverse
Data inverse real data

- Total data
- Total-data inverse
Data inverse real data

estimated primaries

estimated-primaries inverse
An encore: imaging of blended data
Motivation

● **Observation:** *Blended* data acquisition is an instance of *compressive sensing* [F.J.H et. al ‘08]

● *Image directly* in the *simultaneously* acquired data domain

● Imaging conditions associated with adjoint-state methods [Tarantola ‘80s, Plessix, Pratt ’00’s] for the wave equation are based on multi-D correlations of wavefields
  – suffer from finite aperture & source effects
  – contain interferences due to blended acquisition

● Alternative approach based on *wavefield inversion*
Adjoint state or reverse-time methods

- At each depth level multi-D correlation of the monochromatic forward and inverse extrapolated wavefields, $\mathbf{U}$ and $\mathbf{V}$
- Zero-offset image [Berkhout, Claerbout, and others, ‘80s]

$$\delta \mathbf{m} \approx \text{diag} \left( \Re \left( \hat{\mathbf{U}} \hat{\mathbf{V}}^\dagger \right) \right)$$

- Consider deconvolution instead, i.e,

$$\hat{\mathbf{G}} = \Re \left( \hat{\mathbf{U}} \hat{\mathbf{V}}^\dagger \right)$$

- Use wavefield inversion technique
  - improve imaging
  - recover from blended data = compressively subsampled data
Wavefields at 30 Hz [real parts]
Imaging by deconvolution

\[
\begin{align*}
b &= \text{vec} \left( \hat{V}^H \right) \\
A &= \hat{U}^H C_2^H \\
\tilde{x} &= \arg \min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon \\
\tilde{G} &= \text{vec}^{-1} \left( C_2^H \tilde{x} \right)
\end{align*}
\]

- Inversion instead of correlation
- Regularized by 2-D curvelet sparsity promotion
- Example for single layer model at transition
Correlation-based versus wavefield inversion

\[ \hat{G} = \Re \left( \hat{U} \hat{V}^\dagger \right) \]

\[ \tilde{G} = \text{vec}^{-1} \left( C_2^H \tilde{x} \right) \]
Imaging of blended data

\[
\begin{align*}
R &= \left( R^{\Sigma_s} \otimes R^{\Sigma_r} \right) \quad \text{(picking operator)} \\
M &= F_2^* \left( e^{i\theta} \right) F_2 \quad \text{(random encoder)} \\
b &= RM \text{vec} \left( \hat{V} \right) \quad \text{(blended wavefield)} \\
A &= RM \hat{U}^H C_2^H \quad \text{(blended focused 2-D curvelet transform)} \\
\tilde{x} &= \arg\min_x \|x\|_1 \quad \text{s.t.} \quad \|Ax - b\|_2 \leq \epsilon \\
\tilde{G} &= \text{vec}^{-1} \left( C_2^H \tilde{x} \right) \quad \text{(imaged data)}
\end{align*}
\]

with \( \theta = \text{Uniform}([0, 2\pi]) \) random phase rotations.

- CS subsampling after Romberg’s [‘08] random convolution
- Regularized by 2-D curvelet sparsity promotion
- Imaged from source-receiver down-sampling after Fourier-space random phase encoding
Imaging of blended data

\[ \text{vec}^{-1}(RM\text{vec}(\hat{V})) \]

Subsampled V

\[ \text{vec}^{-1}(A^Hb) \]

Image by correlation
Imaging of blended data

\[ \tilde{G} = \text{vec}^{-1} \left( C_2^H \tilde{x} \right) \]

\[ \text{diag} \left( \mathcal{R} \left( \tilde{G} \right) \right) \]

Image by deconvolution

Comparison
Conclusions

- Wavefield inversion is a versatile tool in seismic-data processing & imaging

- Curvelet-domain sparsity is a powerful prior that leads to stable inversions of
  - the primary-matrix operator => improved focusing & recovery
  - the adjoint of the primary-matrix operator => improved multiple prediction
  - the data-matrix operator
  - blended wavefields

- Outlook
  - wavefield predictions with improved spectral and amplitude properties
  - wavefield predictions from blended data
  - sparsity-promoting migration & full waveform inversion
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Further reading

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